Quadratic Forms, Unbounded Self-Adjoint Operators and Self-Adjoint Extensions of the Laplace-Beltrami Operator

Juan Manuel Pérez-Pardo
Universidad Carlos III de Madrid
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Outline

• Introduction to the geometrical description of quantum observables
• Unbounded self-adjoint operators
• Quadratic Forms
• Motivating Example: Laplace-Beltrami operator
• **Quantum Phase Space**
  
  • Projective Hilbert Space \( \mathcal{PH} \)
  
  • Complex Structure
  
  • Riemannian metric
  
  • Symplectic Form
  
  \[ J_{[\Psi]}^2 = -I_{[\Psi]} \]
  
  \[ g_{[\Psi]}(\cdot, \cdot) \]
  
  \[ \omega_{[\Psi]}(\cdot, \cdot) \]

• **Compatibility Conditions**

\[ g_{[\Psi]}(Ju, Jv) = g_{[\Psi]}(u, v) \]

\[ \omega_{[\Psi]}(u, v) = g_{[\Psi]}(Ju, v) \]
• Observables \( \Rightarrow \text{Kählerian functions} \)

\[ f \in C^\infty(\mathcal{P}\mathcal{H}, \mathbb{R}) \quad X_f \text{ Hamiltonian Vector Field} \quad \omega(X_f, \cdot) = df(\cdot) \]

\[ \mathcal{L}_{X_f} \omega = 0 \]

\[ \mathcal{L}_{X_f} g = 0 \]

• \( f \) is a real Kählerian function iff it exists a bounded, self--adjoint, linear operator \( A \) such that

\[ f([\Psi]) = \frac{\langle \Psi, A\Psi \rangle}{\|\Psi\|^2} \]
• **Observables** ⇒ **Kählerian functions**

\[ f \in C^\infty(PH, \mathbb{R}) \quad X_f \text{ Hamiltonian Vector Field} \quad \omega(X_f, \cdot) = df(\cdot) \]

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• Linear Operators on a complex Hilbert space $\mathcal{H}$.

$$T : \mathcal{D}(T) \rightarrow \mathcal{H}$$

• Dense Domains

$$\overline{\mathcal{D}(T)} = \mathcal{H}$$

• Bounded Operator

$$\exists M \text{ s.t. } ||T\Phi|| \leq M||\Phi|| \ \forall \Phi \in \mathcal{D}(T)$$

• Bounded operators are continuous operators

$$\lim_{n \rightarrow \infty} \Phi_n = \Phi \quad \lim_{n \rightarrow \infty} T\Phi_n = T\Phi$$
• Elements of the domain must satisfy

\[ \|T\Phi\| < \infty \]

• The operator can be densely defined. But you can not extend it to the whole Hilbert space

\[ \{\Phi_n\} \text{ is Cauchy } \nleftrightarrow \{T\Phi_n\} \text{ is Cauchy} \]

\[ \mathcal{H} = \mathcal{L}^2[0, 1] \quad P := i \frac{d}{dx} \]

\[ \sqrt{x} \in \mathcal{L}^2[0, 1] \quad \text{but} \quad P(\sqrt{x}) = \frac{i}{2\sqrt{x}} \notin \mathcal{L}^2[0, 1] \]
• **Continuity is not completely lost**

  if \( \{\Phi_n\} \) is Cauchy and \( \{T\Phi_n\} \) is Cauchy

  \[
  \lim \Phi_n = \Phi \quad \text{lim } T\Phi_n =: T\Phi
  \]

• **The graph norm**

  \[
  ||| \cdot ||| = \sqrt{||\cdot||^2 + ||T\cdot||^2}
  \]

• **Closed operator**

  \[
  \overline{\mathcal{D}(T)}|||\cdot||| = \mathcal{D}(T)
  \]
• Extensions

\[ \tilde{T} \text{ extends } T \ (\tilde{T} \supset T) \text{ if} \]

\[ \mathcal{D}(\tilde{T}) \supset \mathcal{D}(T) \]

\[ \tilde{T} \mid_{\mathcal{D}(T)} = T \]

• \(T\) is closable if it has closed extensions

• In general there are infinitely many extensions
• Equivalent for bounded operators

• \( T \text{ is symmetric if } T \subseteq T^\dagger \)

\[ \Rightarrow T \text{ is closable} \]

• \( T \text{ is self-adjoint if } T = T^\dagger \)

\[ \Rightarrow T \text{ is closed} \]

• Symmetric operators may have no self-adjoint extensions
• $T$ is a symmetric differential operator on a Riemannian manifold $\Omega$

• $\mathcal{H} = \mathcal{L}^2(\Omega)$

• Integration by parts

$$\langle \Psi, T\Phi \rangle - \langle T\Psi, \Phi \rangle = \Sigma(\Psi, \Phi)$$

$$\Psi, \Phi \in \mathcal{D}(T^\dagger)$$

• S.a. extensions are in correspondence with maximally isotropic subspaces of $\Sigma(\cdot, \cdot)$ if $T$ has equal deficiency indices

• **Spectral Theorem**

Let $A$ be a self-adjoint operator. Then it exists a spectral decomposition of the identity, $E(\lambda)$, such that:

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda)$$

• **Stone’s Theorem**

$A$ is the generator of a strongly continuous one parameter unitary group, $U_t$, if and only if $A$ is a self-adjoint operator

$$U_t = e^{itA}$$
Kähler isomorphism

A Kähler isomorphism is a smooth diffeomorphism $\Theta : \mathcal{P} \mathcal{H} \to \mathcal{P} \mathcal{H}$, such that

$\Theta^* \omega = \omega \quad \Theta^* g = g$

Because of Stone’s Theorem any self-adjoint operator $A$ (not necessarily bounded) defines a strongly continuous one parameter group of Kähler isomorphisms

$\Theta_t([\Psi]) := [\exp(itA)\Psi]$

If $A$ is bounded $\iff (\Theta_t)_{t \in \mathbb{R}}$ is smooth
• $Q(\Phi) := Q(\Phi, \Phi)$ is a quadratic form with domain $\mathcal{D}(Q)$

• Expectation value functions are quadratic forms

• To every s.a. operator $T$ one can associate a quadratic form $Q_T$

\[
\mathcal{D}(Q_T) = \left\{ \Phi \in \mathcal{H} \left| \int_{\sigma(T)} |\lambda| \langle \Phi, dE_\lambda \Phi \rangle \leq \infty \right. \right\} \\
Q_T(\Phi) = \int_{\sigma(T)} \lambda \langle \Phi, dE_\lambda \Phi \rangle
\]
• Closed Quadratic Form
  • $Q$ is semibounded
    \[ Q(\cdot) \geq -M\|\cdot\|^2 \]
  • $Q$ is complete if $\mathcal{D}(Q)$ is complete in the graph norm
    \[ \|\| \cdot \|\| = \sqrt{(M + 1)\|\cdot\|^2 + Q(\cdot)} \]

• Closable Quadratic Form
• **Representation Theorem of Quadratic Forms.** The following statements are equivalent:

  • $Q$ is the form arising from a semibounded self-adjoint operator

  • $Q : \mathcal{H} \rightarrow \mathbb{R}$ is a lower semicontinuous function.

  $$\liminf_{n \rightarrow \infty} Q(\Phi_n) \geq Q(\Phi)$$

  • The domain $\mathcal{D}$ of the quadratic form is complete with respect to the graph norm

Ref: Davies, E.B. *Spectral Theory and Differential Operators, 1995*
Possible definition for observables

- Real lower semicontinuous functions
  - Lower semibounded
    \[ Q(\cdot) \geq -M \]
  - Satisfy the "parallelogram identity"
    \[ Q(\Psi + \Phi) + Q(\Psi - \Phi) = 2 \left( Q(\Psi) + Q(\Phi) \right) \]
Closable Operator

- Symmetric Operators are always closable
- The minimal extension doesn’t need to be s.a. operator
- It is possible that none of the extensions is s.a.

Closable Quadratic Forms

- Hermitian quadratic forms are not always closable
- If they are closable the minimal extension is always associated with a s.a. Operator (Friedrichs’ extension)
Advantages of the Quadratic Forms

- No information is required about the adjoint operator
- It is not necessary to compute the deficiency indices
- They contain information of the spectrum of the operator
  - min-max Principle
- They are more suitable for numerics
  - They are explicitly Hermitian
  - Their domains are bigger $\mathcal{D}(Q_T) \supset \mathcal{D}(T)$
• Laplace-Beltrami operator on a compact Riemannian manifold \((\Omega, \eta)\)

\[
\Delta_\eta = \sum_{j,k} \frac{1}{\sqrt{|\eta|}} \frac{\partial}{\partial x^j} \sqrt{|\eta|} \eta^{jk} \frac{\partial}{\partial x^k}
\]

• Single integration by parts leads to

\[
\langle \Phi, -\Delta_\eta \Phi \rangle = \langle d\Phi, d\Phi \rangle - \langle \varphi, \dot{\varphi} \rangle_{L^2(\partial \Omega)} =: Q(\Phi)
\]

\[
\Phi \in C^\infty(\Omega)
\]

\[
\varphi := \Phi \big|_{\partial \Omega} \quad \dot{\varphi} := \frac{d\Phi}{dn} \big|_{\partial \Omega}
\]
• Hermitian Domains for $Q$
  \[ \langle \varphi, \dot{\varphi} \rangle \neq \langle \dot{\varphi}, \varphi \rangle \]

Maximally Isotropic Subspaces of
  \[ \Sigma(\varphi, \dot{\varphi}) := \langle \varphi, \dot{\varphi} \rangle - \langle \dot{\varphi}, \varphi \rangle \]

• Cayley Transform \( \varphi_{\pm} = \varphi \pm i\dot{\varphi} \)
  \[ \Sigma(\varphi, \dot{\varphi}) = \langle \varphi, \dot{\varphi} \rangle - \langle \dot{\varphi}, \varphi \rangle \]
  \[ \downarrow \]
  \[ \tilde{\Sigma}(\varphi_+, \varphi_-) = i[\langle \varphi_+, \varphi_+ \rangle - \langle \varphi_-, \varphi_- \rangle] \]
  \[ \varphi_- = U \varphi_+, \quad U \in \mathcal{U}(L^2(\partial\Omega)) \]
  \[ \varphi - i\dot{\varphi} = U(\varphi + i\dot{\varphi}) \]
Towards the semibounded condition

- **Semiboundedness**

Assume that \( \dist(\sigma(U)/\{-1\}, -1) \geq \kappa > 0 \Rightarrow U = \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix} \)

\( \langle \varphi, \dot{\varphi} \rangle = \langle \varphi_0, \dot{\varphi}_0 \rangle + \langle \varphi_1, \dot{\varphi}_1 \rangle \)

- **Completeness**
• **Semiboundedness**

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\[
\langle \varphi, \dot{\varphi} \rangle = \langle \varphi_0, \dot{\varphi}_0 \rangle + \langle \varphi_1, \dot{\varphi}_1 \rangle \leq ||A_1|| ||\varphi||^2
\]

• **Completeness**
Towards the semibounded condition

- **Semiboundedness**

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\langle \varphi, \dot{\varphi} \rangle = \langle \varphi_0, \dot{\varphi}_0 \rangle + \langle \varphi_1, \dot{\varphi}_1 \rangle \leq \|A_1\| \|\varphi\|^2
\]

\[
\text{Trace inequality: } \|\varphi\| \leq C \|\Phi\|_{\mathcal{H}^1}
\]

- **Completeness**
• **Semiboundedness**

Assume that \( \text{dist}(\sigma(U)/\{-1\}, -1) \geq \kappa > 0 \) \( \Rightarrow \) \( U = \begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix} \)

\[
\langle \varphi, \dot{\varphi} \rangle = \langle \varphi_0, \dot{\varphi}_0 \rangle + \langle \varphi_1, \dot{\varphi}_1 \rangle \leq \|A_1\| \|\varphi\|^2
\]

**Trace inequality:** \( \|\varphi\| \leq C\|\Phi\|_{\mathcal{H}^1} \)

• **Completeness**

\[
\|\| \cdot \| \| = \sqrt{(M + 1)\| \cdot \|^2 + Q(\cdot)} \sim \| \cdot \|_{\mathcal{H}^1}
\]
The quadratic forms

\[ Q_U := \langle d\Phi, d\Phi \rangle - \langle \varphi, \dot{\varphi} \rangle \]

\[ \mathcal{D}(Q_U) = \{ \Phi \in \mathcal{C}^\infty(\Omega) | \varphi - i\dot{\varphi} = U(\varphi + i\dot{\varphi}) \} \]

- Are **closable** provided that
  - \[ |\langle \varphi, \dot{\varphi} \rangle| \leq M||\Phi||^2_{\mathcal{H}^1} \quad M < 1 \]

- The domains of their closures are

\[ \overline{\mathcal{D}(Q_U)}^{\mathcal{H}^1(\Omega)} \]

- They are quadratic forms associated to s.a. extensions of the Laplace-Beltrami Operator

• There are self-adjoint extensions of the Laplace-Beltrami operator that are not semibounded.

• They still define quadratic forms

• Another characterization of completeness that does not use semiboundedness

The quadratic form $Q$ is complete iff whenever a sequence $\{\Phi_n\}$ satisfies $\lim \Phi_n = \Phi$ and $Q(\Phi_n - \Phi_m) \to 0$ as $n, m \to \infty$, then $\Phi \in \mathcal{D}$ and $Q(\Phi - \Phi_n) \to 0$.

Conclusions and further work

• Even for standard Q.M. one can take advantage of the properties of the quadratic forms. For instance, they contain the information about the spectrum of the operator.

• Quadratic Forms arise as the natural objects to define observables in the Geometrical Picture. However, a characterization for the infinite dimensional case is still missing. A generalized version of the representation theorem that accounts for quadratic forms that are not lower semibounded, but satisfy a weaker condition, might be the guide for a proper characterization.
Thanks for your attention